OSCILLATORY PERTURBATIONS IN A CONDUCTING FLUID IN A MAGNETIC FIELD

(OSTSILLIRUIUSHCHIE VOZMUSHCHENIIA V PROVODIASHCHEI ZHIDKOSTI V MAGNITNOM POLE)

PMM Vol.27, No.3, 1963, pp. 523-531

M.I. SHLIOMIS
(Ivanovo)

(Received December 12, 1962)

This paper studies in a general fashion the influence of a homogeneous magnetic field on the character of the decay of small perturbations in the equilibrium of a conducting fluid in a cavity of arbitrary shape. It is shown that for small values of the Hartmann number M there exist two types of normal perturbation: "magnetic" and "hydrodynamic". Perturbations of both types always decay monotonically, Above a certain critical value of $M = M_*$ there exist in the spectrum of normal perturbations some which can no longer be referred to as magnetic or hydrodynamic: they cannot be classified thus. These normal perturbations oscillate with respect to time; moreover, for values of M not greatly exceeding M_*, their frequency is proportional to $(M - M_*)^{1/2}$. The occurrence of the critical point is essentially connected with the presence of two types of perturbation for small M. This paper fully investigates the nature of the singularity of perturbations at this particular point.

1. The equations for normal perturbations. In the external homogeneous magnetic field

$$\mathbf{H}_0 = \gamma H_0, \qquad \gamma^2 = 1 \tag{1.1}$$

The fluid with conductivity σ fills a cavity of arbitrary shape cut out of an infinitely hard solid with conductivity σ°.

In equilibrium the velocity of the fluid is everywhere equal to zero and the magnetic field is equal to the external field.

For small perturbations of the equilibrium the fluid moves with velocity $\mathbf{u}$ and a perturbation magnetic field $\mathbf{h}$ appears (in the surrounding material as well as in the fluid). The linearized equations for the perturbations are [1]

$$\text{in the fluid} \quad \mathbf{u} = -\nabla p + \nabla^2\mathbf{u} + M(\gamma\nabla)\mathbf{h}, \quad \nabla\mathbf{u} = 0$$
$$N\dot{\mathbf{h}} = \nabla^2\mathbf{h} + M(\gamma\nabla)\mathbf{u}, \quad \nabla\mathbf{h} = 0 \tag{1.2}$$

$$\text{in the solid} \quad N\dot{\mathbf{h}} = \frac{\sigma}{\sigma^\circ}\nabla^2\mathbf{h}, \qquad \nabla\mathbf{h} = 0 \tag{1.3}$$

In these equations the following dimensionless parameters occur:

$$N = \frac{4\pi\eta\sigma}{\rho c^2}, \qquad M^2 = \frac{H_0^2\sigma l^2}{\eta c^2} \tag{1.4}$$

Here l is a characteristic dimension of the cavity, ρ and η are the density and viscosity of the fluid. Assuming N to be constant, we shall investigate equations (1.2) as to the manner of their dependence on the Hartmann number M.

At the boundary of the cavity the velocity of the fluid must vanish, there must be continuity of the magnetic field, and moreover [2], there must be continuity of the component of the electric field tangential to the boundary of the cavity. Hence follow the boundary conditions at the surface of the cavity:

$$\mathbf{u} = 0, \qquad \mathbf{h} = \mathbf{h}^\circ, \qquad (\operatorname{rot}\mathbf{h})_t = \frac{\sigma}{\sigma^\circ}(\operatorname{rot}\mathbf{h}^\circ)_t \tag{1.5}$$

(We denote by the small superscript circle the values of quantities in the solid.) At infinity $\mathbf{h} \to 0$. It is easy to verify that, by virtue of the boundary conditions, Gauss' theorem implies, for example, that

$$\int_V \mathbf{h}^*\cdot\nabla^2\mathbf{h}\,dV + \frac{\sigma}{\sigma^\circ}\int_{V^\circ}\mathbf{h}^*\cdot\nabla^2\mathbf{h}\,dV = -\int_V \operatorname{rot}\mathbf{h}^*\cdot\operatorname{rot}\mathbf{h}\,dV -$$
$$-\frac{\sigma}{\sigma^\circ}\int_{V^\circ}\operatorname{rot}\mathbf{h}^*\cdot\operatorname{rot}\mathbf{h}\,dV = \int_V \mathbf{h}\cdot\nabla^2\mathbf{h}^*\,dV + \frac{\sigma}{\sigma^\circ}\int_{V^\circ}\mathbf{h}\cdot\nabla^2\mathbf{h}^*\,dV$$

and similarly in other cases. For brevity we shall write

$$\int \text{ in place of } \int_V + \frac{\sigma}{\sigma^\circ}\int_{V^\circ} \tag{1.6}$$

For brevity of notation we shall introduce the six-component perturbation u, the operators K and I acting on u, and the gradient of "pressure" ∇p

$$u \equiv \begin{bmatrix} \mathbf{u} \\ \mathbf{h} \end{bmatrix}, \qquad K = \begin{bmatrix} 1 & 0 \\ 0 & N \end{bmatrix}, \qquad I = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \qquad \nabla p \equiv \begin{bmatrix} \nabla p \\ 0 \end{bmatrix} \quad (1.7)$$

Then equations (1.2) can be written thus:

$$K\dot{u} = -\nabla p + \nabla^2 u + M(\gamma\cdot\nabla) I u, \qquad \nabla u = 0 \quad (1.8)$$

The quantity which is the Hermitian conjugate of (1.7) will be denoted by

$$u^+ \equiv [\mathbf{u}^*, \mathbf{h}^*] \quad (1.9)$$

and the scalar product of v with u will be the integral

$$(v\cdot u) \equiv \int v^+ u dV = \int [\mathbf{v}^*, \ \mathbf{g}^*]\cdot \begin{bmatrix} \mathbf{u} \\ \mathbf{h} \end{bmatrix} dV = \int \{\mathbf{v}^*\cdot\mathbf{u} + \mathbf{g}^*\cdot\mathbf{h}\}\, dV \quad (1.10)$$

If we set $u \sim e^{-\lambda t}$, then from (1.8) we obtain the equation for the normal perturbations

$$-\lambda K u = L u \equiv -\nabla p + \nabla^2 u + M(\gamma\cdot\nabla) I u, \qquad \nabla u = 0 \quad (1.11)$$

The operator L in (1.11) is not self-conjugate. In fact, forming the scalar product of the right-hand side of (1.11) with v and applying Gauss' theorem, we obtain

$$(v\cdot\{-\nabla p + \nabla^2 u + M(\gamma\cdot\nabla) I u\}) = (\{-\nabla q + \nabla^2 v - M(\gamma\cdot\nabla) I v\}\cdot u)$$

(with a "minus" sign in front of M) so that the conjugate equations of (1.11) are

$$-\lambda^* K v = L^+ v \equiv -\nabla q + \nabla^2 v - M(\gamma\nabla) I v, \qquad \nabla v = 0 \quad (1.12)$$

The fact that $L^+(M)$ is equal simply to $L(-M)$ enables us to conclude that if u, defined by (1.7), is the solution of equation (1.11), then (1.12) has the solution

$$v = \begin{bmatrix} \mathbf{u}^* \\ -\mathbf{h}^* \end{bmatrix} \quad (1.13)$$

The solution of the boundary value problem (1.11) or (1.12) with the boundary conditions (1.5) gives an infinite sequence of eigenvalues or decrements λ_α and the corresponding normal perturbations u_α; simultaneously we also obtain a sequence of conjugate normal perturbations v_α with decrements λ_α^*. As was shown above

$$v_\alpha = \begin{bmatrix} \mathbf{v}_\alpha \\ \mathbf{g}_\alpha \end{bmatrix} = \begin{bmatrix} \mathbf{u}_\alpha{}^* \\ -\mathbf{h}_\alpha{}^* \end{bmatrix} \tag{1.14}$$

All the decrements λ_α may turn out to be "simple", and then the system of quantities u_α is complete, i.e. any quantity u can be expanded as a series in the u_α. The case of degeneracy (multiple decrements) is more complicated, and evidently, as will appear below, it can also be of great interest.

It is easy to see that each normal perturbation with decrement λ_α is orthogonal in a certain sense to all conjugate normal perturbations whose decrements $\lambda_\beta{}^*$ are not equal to $\lambda_\alpha{}^*$. In fact, from (1.11) and (1.12) we obtain

$$-\lambda_\beta\,(v_\alpha \cdot K u_\beta) = (v_\alpha \cdot L u_\beta) = (L^+ v_\alpha \cdot u_\beta) = -\lambda_\alpha\,(K v_\alpha \cdot u_\beta) = -\lambda_\alpha\,(v_\alpha \cdot K u_\beta)$$

and consequently, for $\lambda_\alpha \neq \lambda_\beta$

$$(v_\alpha \cdot K u_\beta) = \int \{\mathbf{u}_\alpha \cdot \mathbf{u}_\beta - N \mathbf{h}_\alpha \cdot \mathbf{h}_\beta\}\, dV = 0 \tag{1.15}$$

Finally, from (1.11) it is easy to obtain the integral relations

$$(\lambda + \lambda^*) \int \{\mathbf{u}^* \cdot \mathbf{u} + N \mathbf{h}^* \mathbf{h}\}\, dV = \int \{\mathrm{rot}\ \mathbf{u}^* \cdot \mathrm{rot}\ \mathbf{u} + \mathrm{rot}\ \mathbf{h}^* \cdot \mathrm{rot}\ \mathbf{h}\}\, dV \tag{1.16}$$

$$(\lambda - \lambda^*) \int \{\mathbf{u}^* \cdot \mathbf{u} - N \mathbf{h}^* \cdot \mathbf{h}\}\, dV = 0 \tag{1.17}$$

From these it follows, firstly, that $\mathrm{Re}\,\lambda > 0$, i.e. that all normal perturbations decay with time and, secondly, that a necessary condition for the existence of multiple decrements (i.e. oscillating perturbations) is the vanishing of the integral in (1.17).

Suppose that there exists a Hartmann number M_* such that a certain normal perturbation of the number α decays monotonically ($\mathrm{Im}\,\lambda_\alpha = 0$) when $M < M_*$, and oscillates with time when $M > M_*$. In the latter range of the Hartmann number, as follows from (1.17), the quantity u_α is orthogonal to its complex conjugate $u_\alpha{}^*$:

$$\int \{\mathbf{u}_\alpha{}^* \cdot \mathbf{u}_\alpha - N \mathbf{h}_\alpha{}^* \cdot \mathbf{h}_\alpha\}\, dV = 0 \tag{1.18}$$

This equation, valid for $M > M_*$, must by continuity be fulfilled also at the point M_*, where the integral (1.18) coincides with the normalizing integral (since $\mathrm{Im}\,\lambda_\alpha(M_*) = 0$, then the eigenvector u_α can be chosen to be real: $u_\alpha{}^* = u_\alpha$). Accordingly, if the Hartmann number increases initially from zero (when $M = 0$ all the perturbations decay monotonically), then the appearance of each new oscillating perturbation

α each time will precede the vanishing of the corresponding normalizing integral

$$(v_\alpha\ \{M_*\}\cdot Ku_\alpha\ \{M_*\}) = \int \{\mathbf{u}_\alpha^2 - N\mathbf{h}_\alpha^2\}\, dV = 0 \tag{1.19}$$

Clearly, this case needs investigating.

2. The two types of normal perturbations for small Hartmann numbers. If there is no external field, then $M = 0$, and equations (1.11) can be separated into the two unconnected problems

$$-\lambda\mathbf{u} = \nabla^2\mathbf{u} - \nabla p, \qquad \nabla\mathbf{u} = 0; \qquad\qquad -\lambda N\mathbf{h} = \nabla^2\mathbf{h}, \qquad \nabla\mathbf{h} = 0 \tag{2.1}$$

Solving these, we obtain an infinite sequence of magnetic perturbations $\lambda_{1\alpha}$, $u_{1\alpha}$ and another infinite sequence of hydrodynamic perturbations $\lambda_{2\alpha}$, $u_{2\alpha}$

$$\lambda_{1\alpha}, \quad u_{1\alpha} = \begin{bmatrix} 0 \\ \mathbf{h}_{1\alpha} \end{bmatrix}, \qquad \lambda_{2\alpha}, \quad u_{2\alpha} = \begin{bmatrix} \mathbf{u}_{2\alpha} \\ 0 \end{bmatrix} \qquad (\alpha = 0, 1, 2, \ldots) \tag{2.2}$$

With an external field each motion of the fluid will be accompanied by a perturbation of the magnetic field, whilst each perturbation of the field will be accompanied by a motion of the fluid, so that the type of perturbation is not immediately obvious. However, for sufficiently small values of M, simply by continuity, we can speak conventionally of "magnetic" and "hydrodynamic" perturbations, and we can establish a perfectly precise criterion to distinguish between them. Indeed, as was shown in [1] (where, however, the existence of two types of perturbation was not remarked upon), for small values of M there exist expansions of the normal perturbations as series in M^2, In [1], series of the following forms were obtained:

$$\mathbf{u}_2 = \mathbf{u}_2^{(0)} + M^2\mathbf{u}_2^{(1)} + M^4\mathbf{u}_2^{(2)} + \ldots, \qquad \mathbf{h}_2 = M\mathbf{h}_2^{(1)} + M^3\mathbf{h}_2^{(2)} + \ldots$$
$$\lambda_2 = \lambda_2^{(0)} + M^2\lambda_2^{(1)} + M^4\lambda_2^{(2)} + \ldots \tag{2.3}$$

When $M = 0$ only the velocity remains, whilst the magnetic field vanishes, so that the perturbations represented by these series are continuous extensions of the hydrodynamic perturbations. It is clear that in their case, at least for sufficiently small values of M, the normalizing integrals are positive

$$\int \{\mathbf{u}_2^2 - N\mathbf{h}_2^2\}\, dV > 0 \tag{2.4}$$

From the symmetry of equations (1.11) with respect to $\mathbf{u}$ and $\mathbf{h}$ it follows at once that there will also exist other expansions

$$\mathbf{u}_1 = M\mathbf{u}_1^{(1)} + M^3\mathbf{u}_1^{(2)} + \ldots, \qquad \mathbf{h}_1 = \mathbf{h}_1^{(0)} + M^2\mathbf{h}_1^{(1)} + M^4\mathbf{h}_1^{(2)} + \ldots$$

$$\lambda_1 = \lambda_1^{(0)} + M^2\lambda_1^{(1)} + M^4\lambda_1^{(2)} + \ldots \tag{2.5}$$

The perturbations represented by them tend continuously, as $M \to 0$, to magnetic perturbations and it is clear that for sufficiently small values of M their normalizing integrals are negative

$$\int \{\mathbf{u}_1^2 - N\mathbf{h}_1^2\}\, dV < 0 \tag{2.6}$$

It can be shown, as was done in [1], that all the expansions (2.3) and (2.5) are real, so that as long as they converge there are no oscillatory perturbations. The perturbations moreover can be classified unambiguously into "magnetic" (first subscript *1*) or "hydrodynamic" (first subscript 2). As has already been shown, they are orthogonal among themselves and can be normalized, so that

$$(v_{m\alpha} \cdot Ku_{n\beta}) \equiv \int \{\mathbf{u}_{m\alpha} \cdot \mathbf{u}_{n\beta} - N\mathbf{h}_{m\alpha} \cdot \mathbf{h}_{n\beta}\}\, dV = (-)^n\, \delta_{mn}\delta_{\alpha\beta}$$

$$(m,\ n = 1,\ 2;\ \alpha = 0,\ 1,\ 2,\ \ldots) \tag{2.7}$$

The situation described above will persist up to the point when, for a certain $M = M_*$, the normalizing integral of any normal perturbation does not vanish. The decrements cannot then remain simple. Indeed, then all the normal perturbations at M_* would be orthogonal to one another as before, whilst one of them (that for which the normalizing integral is zero) would still be orthogonal even to itself. The system of normal perturbations consequently ceases to be complete.* But this is improbable on physical grounds, since any small perturbation must be composed of the normal perturbations described by equations (1.11).

It is evidently necessary to investigate the point of confluence of the two decrements. Here the following two cases may occur:

---

* It is easy to see that if for $u_{m\alpha}$

$$(v_{m\alpha} \cdot Ku_{m\alpha}) = 0$$

then this quantity cannot be expanded in terms of $u_{n\beta}$; all the coefficients $a_{n\beta}$ of the series

$$u_{m\alpha} = \sum_{n,\ \beta}{}' a_{n\beta}u_{n\beta}$$

turn out to be zeros.

1. At the point of confluence M_* there exist two eigen solutions of the problem (1.11), i.e. two normal perturbations. The requirement formulated above concerning completeness of the system of normal perturbations enables us to conclude that in this case neither one of the normalizing integrals can vanish at the point M_*.

2. At the point of confluence of two decrements there do not exist two normal perturbations. This case is studied in the following section.

3. Branch points. The fact that the operator L in problem (1.11) is not self-conjugate has already been mentioned in Section 1. It is well known [3] that the number of eigenfunctions of such an operator, corresponding to a degenerate eigenvalue, may be less than the degree of degeneracy. For the given problem it is sufficient to consider the case of multiplicity two, when the operator L transforms two linearly independent quantities u_1 and u_2 into linear combinations of themselves. One of them can always be chosen so that it is an eigenfunction for L:

$$Lu_1 = -\lambda K u_1, \qquad Lu_2 = -\mu K u_2 + \nu K u_1 \tag{3.1}$$

If $\mu \neq \lambda$, then it is easy to see that a certain linear combination of u_1 and u_2 is a second eigenfunction for L. Accordingly, for a λ with degeneracy of degree two, there are two functions (the first of them is called the eigenfunction, and the second the associated function), for which

$$Lu_1 = -\lambda K u_1, \qquad Lu_2 = -\lambda K u_2 + K u_1 \tag{3.2}$$

(By changing the normalization of u_1, we can always make $\nu = 1$.)

It proves to be more convenient, in place of the eigenfunction and the associated function, to consider their sum and their difference, which we shall denote as before by u_1 and u_2. Obviously for them we have the equations

$$Lu_1 = -\lambda K u_1 + \tfrac{1}{2} K (u_1 + u_2), \quad Lu_2 = -\lambda K u_2 - \tfrac{1}{2} K (u_1 + u_2) \tag{3.3}$$

and for their conjugates

$$L^+ v_1 = -\lambda^* K v_1 + \tfrac{1}{2} K (v_1 + v_2), \; L^+ v_2 = -\lambda^* K v_2 - \tfrac{1}{2} K (v_1 + v_2) \tag{3.4}$$

From (3.3) and (3.4) it follows that

$$(v_1 \cdot K u_2) = (v_2 \cdot K u_1), \qquad (v_1 \cdot K u_1) + (v_2 \cdot K u_2) + 2\,(v_1 \cdot K u_2) = 0 \tag{3.5}$$

Without affecting u_1 and u_2, we can replace v_1 and v_2 by linear combinations of them

$$v_1 \to av_1 + bv_2, \qquad v_2 \to -bv_1 + (a-2b)\, v_2 \tag{3.6}$$

(where a and b are arbitrary), again satisfying the equations (3.4). The constants a and b can be selected so that

$$(v_1 \cdot Ku_2) = 0 \tag{3.7}$$

and then from (3.5) it follows that

$$(v_1 \cdot Ku_1) = -(v_2 \cdot Ku_2) \tag{3.8}$$

In the latter notation the sum $u_1 + u_2$ is the eigenfunction of the operator L, whilst $v_1 + v_2$ is that of the operator L^+. Taking into consideration (3.7) and (3.8), we find that the normalizing integral for the eigenfunction is equal to zero

$$((v_1 + v_2) \cdot K\, (u_1 + u_2)) = 0 \tag{3.9}$$

Accordingly, the vanishing of the normalizing integral is necessarily connected with the confluence of two eigenvalues and the disappearance of an eigenfunction. The direction of further investigation is now clear.

Let us suppose that at the point $M = M_*$ all the decrements are real and all except one, namely λ_*, are simple. The multiplicity of λ_* is equal to two. To this decrement there correspond two "quasinormal" perturbations, satisfying equations of the form (3.3)

$$\begin{aligned} -\lambda_* Ku_{10} &= -\nabla p_{10} + \nabla^2 u_{10} + M_* (\gamma \cdot \nabla)\, Iu_{10} + \tfrac{1}{2} K\, (u_{10} + u_{20}) \\ -\lambda_* Ku_{20} &= -\nabla p_{20} + \nabla^2 u_{20} + M_* (\gamma \cdot \nabla)\, Iu_{20} - \tfrac{1}{2} K\, (u_{10} + u_{20}) \end{aligned} \tag{3.10}$$

The conjugate quasinormal perturbations, satisfying the same equations, but with M_* replaced by $-M_*$, will be chosen so that

$$(v_{m0}, Ku_{n0}) = (-)^n \delta_{mn} \qquad (m, n = 1, 2) \tag{3.11}$$

It is easy to see that the time-dependent solutions of equation (1.8), corresponding to λ_*, are

$$e^{-\lambda_* t}\, (u_{10} + u_{20}), \qquad e^{-\lambda_* t}\, (u_{10} - u_{20}) + te^{-\lambda_* t}\, (u_{10} + u_{20}) \tag{3.12}$$

To the simple decrements $\lambda_{n\alpha} (n = 1, 2;\ \alpha > 0)$ there correspond the normal perturbations $u_{n\alpha}$, for which

$$-\lambda_{n\alpha} K u_{n\alpha} = -\nabla p_{n\alpha} + \nabla^2 u_{n\alpha} + M_* (\gamma \cdot \nabla) I u_{n\alpha} \tag{3.13}$$

All the normal and quasinormal perturbations form a complete system of functions

$$\lambda_*, \quad u_{10}, \quad u_{20}, \qquad \lambda_{n\alpha}, \quad u_{n\alpha} \quad (n = 1, 2; \ \alpha = 1, 2, \ldots)$$

mutually orthogonal and normalized according to the condition (2.7). Thus, any function u can be expanded in a series of the form

$$u = \sum_{n, \alpha} b_{n\alpha} u_{n\alpha}, \quad b_{n\alpha} = (-1)^n (v_{n\alpha}, Ku) \quad (n = 1, 2; \ \alpha = 0, 1, 2, \ldots) \tag{3.14}$$

In what follows the second index (zero) for the quasinormal perturbations will be dropped.

Let us now consider values of M close to M_*. For these values there does not exist an expansion of u and λ in integral powers of $\xi = (M - M_*)$, i.e. the point M_* is singular. In order to become convinced of this, let us rewrite equation (1.11) in the form

$$-\lambda K u = -\nabla p + \nabla^2 u + M_* (\gamma \nabla) I u + \xi (\gamma \nabla) I u \tag{3.15}$$

and in it set

$$u = u^{(0)} + \xi u^{(1)} + \ldots, \qquad \lambda = \lambda_* + \xi \lambda^{(1)} + \ldots \tag{3.16}$$

Multiplying (3.15) by $v_1 + v_2$ and collecting terms not containing ξ we obtain, by virtue of (3.10), $u^{(0)} \sim (u_1 + u_2)$. From a consideration of the terms containing the first power of ξ it follows that

$$((v_1 + v_2) \cdot (\gamma \cdot \nabla) I (u_1 + u_2)) \equiv \int (\mathbf{u}_1 + \mathbf{u}_2) (\gamma \cdot \nabla) (\mathbf{h}_1 + \mathbf{h}_2) \, dV = 0 \tag{3.17}$$

But the integral in (3.17) can vanish only by pure chance. In fact it is not difficult to see from (1.11) and (2.1) that this would indicate that when $M = 0$ two perturbations of different types have the same decrement

$$\lambda_{1\alpha}(0) = \lambda_{2\alpha}(0) = \lambda_* (M_*)$$

For such a coincidence there is, of course, no physical basis at all: in the absence of an external field the magnetic perturbations are in no way connected with the hydrodynamic ones.

We shall show that the singular point M_* is a branch point, about which the degenerate decrements and the normal perturbations can be expanded in integral powers of the quantity

$$\eta = (M - M_*)^{1/2} \tag{3.18}$$

i.e. there exist series of the form

$$u = u^{(0)} + \eta u^{(1)} + \eta^2 u^{(2)} + \ldots, \qquad \lambda = \lambda_* + \eta\lambda^{(1)} + \eta^2\lambda^{(2)} + \ldots \tag{3.19}$$

satisfying equations (1.11), and corresponding series for equations (1.12). The substitution of these series in the equations and the collection of terms with the same powers of η gives a sequence of equations which we shall not write down in general form. The equation of zero order is

$$-\lambda_* K u^{(0)} = -\nabla p^{(0)} + \nabla^2 u^{(0)} + M_* (\gamma\nabla) I u^{(0)} \tag{3.20}$$

From this it follows (see (3.10)) that

$$u^{(0)} = u_1 + u_2 \tag{3.21}$$

This could have been foreseen, moreover, since only the sum of the quasinormal perturbations changes with time according to the purely exponential law, which governs the normal perturbations when $M < M_*$. In order that all the $u^{(k)}$ in the expansion (3.19) shall not contain $(u_1 + u_2)$, we need an equivalent change of the normalization. Then

$$u^{(k)} = b^{(k)} (u_1 - u_2) + \sum_{n,\, \alpha>0} b_{n\alpha}^{(k)} u_{n\alpha} \tag{3.22}$$

and consequently

$$(v_1 \cdot K u^{(k)}) = (v_2 \cdot K u^{(k)}) = -b^{(k)} \tag{3.23}$$

Let us now consider the first order equation

$$-\lambda_* K u^{(1)} + \nabla p^{(1)} - \nabla^2 u^{(1)} - M_* (\gamma \cdot \nabla) I u^{(1)} = \lambda^{(1)} K (u_1 + u_2) \tag{3.24}$$

Taking the scalar product of this with v_1 (or v_2), we obtain

$$(K (v_1 + v_2) \cdot u^{(1)}) = -2\lambda^{(1)}$$

i.e. according to (3.23)

$$b^{(1)} = \lambda^{(1)} \tag{3.25}$$

whilst, multiplying by $v_{n\alpha}$ $(\alpha > 0)$

$$(v_{n\alpha} \cdot K u^{(1)}) = b_{n\alpha}^{(1)} = 0 \tag{3.26}$$

Consequently

$$u^{(1)} = \lambda^{(1)} (u_1 - u_2) \tag{3.27}$$

where $\lambda^{(1)}$ remains indeterminate.

The second order gives

$$\begin{aligned} &- \lambda_* K u^{(2)} + \nabla p^{(2)} - \nabla^2 u^{(2)} - M_* (\gamma \cdot \nabla) I u^{(2)} = \\ &= \lambda^{(1)} K u^{(1)} + \lambda^{(2)} K (u_1 + u_2) + (\gamma \cdot \nabla) I (u_1 + u_2) \end{aligned} \tag{3.28}$$

Taking the scalar product of this with $(v_1 + v_2)$, we obtain

$$[\lambda^{(1)}]^2 = ((v_1 + v_2) \cdot (\gamma \cdot \nabla) I (u_1 + u_2)) \equiv - B^2 \tag{3.29}$$

whilst multiplying by $(v_1 - v_2)$, we get

$$b^{(2)} = \lambda^{(2)} - ((v_1 - v_2) \cdot (\gamma \cdot \nabla) I (u_1 + u_2))$$

Multiplying equation (3.28) by $v_{n\alpha}$ would give $b_{n\alpha}{}^{(2)}$ and so on. It is evident that in this manner we can obtain all the coefficients of the series (3.19).

When $M < M_*$, i.e. for purely imaginary

$$\eta = i\varepsilon \qquad (\varepsilon > 0) \tag{3.30}$$

according to our assumption we must obtain real normal perturbations, corresponding to two different real decrements.

Their expansions, as now determined (formulas (3.21), (3.27) and (3.29)), have the forms

$$\begin{aligned} \lambda_{10} (M_* - \varepsilon^2) &= \lambda_* + \varepsilon B + \ldots, \\ u_{10} (M_* - \varepsilon^2) &= (u_1 + u_2) + \varepsilon B (u_1 - u_2) + \ldots \end{aligned} \tag{3.31}$$

$$\lambda_{20} = \lambda_* - \varepsilon B + \ldots, \qquad u_{20} = (u_1 + u_2) - \varepsilon B (u_1 - u_2) + \ldots \tag{3.32}$$

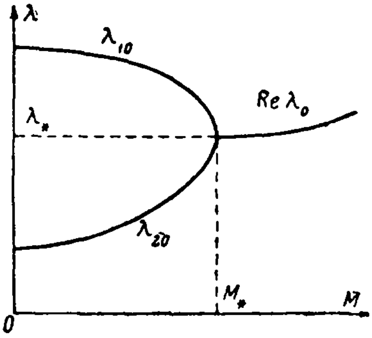
Accordingly, B is real, i.e. the integral

$$((v_1 + v_2) \cdot (\gamma \cdot \nabla) I (u_1 + u_2)) < 0 \tag{3.33}$$

Choosing $B > 0$, from (3.31) and (3.32) we find that

$$(v_{10} \cdot K u_{10}) = - 2B\varepsilon + \ldots < 0, \qquad (v_{20} \cdot K u_{20}) = 2B\varepsilon + \ldots > 0 \tag{3.34}$$

This leads to an important result: the critical point is the confluence of decrements of essentially different types - magnetic and hydrodynamic (see figure).



On the other side of the branch point, i.e. when $M > M_*$ η is real and we obtain two complex conjugate decrements with corresponding normal perturbations. Their expansions close to M_* are

$$\lambda_{(1,2)\,0} = \lambda_* \pm iB\eta + \ldots, \qquad (3.35)$$

$$u_{(1,2)\,0} = (u_1 + u_2) \pm iB\eta\,(u_1 - u_2) + \ldots$$

so that their variation with time is oscillatory with frequency given by

$$B\eta = B\,(M - M_*)^{1/2} \qquad (3.36)$$

I offer my profound thanks to V.S. Sorokin for suggesting the theme of the present paper, for his attentive advice and valuable assistance.

BIBLIOGRAPHY

1. Sorokin, V.S. and Sushkin, I.V., Ustoichivost' ravnovesiia podogrevaemoi snizu provodiashchei zhidkosti v magnitnom (Stability of equilibrium of a conducting fluid heated from below in a magnetic field). *ZhETF*, Vol. 38, No. 2, 1960.

2. Landau, L.D. and Lifshitz, E.M., *Elektrodinamika sploshnykh sred (Electrodynamics of continuous media)*. GITTL, 1957.

3. Keldysh, M.V., O sobstvennykh znacheniiakh i sobstvennykh funktsiiakh nekotorykh klassov nesamosopriiazhennykh uravnenii (On the eigenvalues and eigenfunctions of certain classes of non-self-conjugate equations). *Dokl. Akad. Nauk SSSR*, Vol. 77, No. 1, 1957.

Translated by A.H.A.